

My favorite theorems by Jeff Hirst

Peter Cholak

August, 2025

Hirstfest

<https://academicweb.nd.edu/~cholak/hirst25.pdf>

Abstract: Join me for a look at my personal selection of Jeff Hirst's most impactful theorems. Beyond mathematics, I'll share stories and insights from over 30 years of friendship and intellectual exchange with Jeff, offering a unique perspective on his work and our time together.

Jeff (CT?), enjoy your (self-defined)
retirement!

Jeff's big lessons:

- Know your domain of discourse!
- It matters how you say something.

My distillation of these values

- Try be to careful where one works.
- Try to be careful about the details one ignores.
- Be sure to know actually know the details, or someone who does, or, at least, how to redirect if pressed.

Enough . . . get to the actual theorems!

Theorem (Hirst)

Over RCA_0 ,

- $RT_{<\infty}^1$ is equivalent to $B\Sigma_2$.
- RT_2^2 implies $RT_\infty^1 \equiv B\Sigma_2$.
- $RT_{<\infty}^2$ implies $B\Sigma_3$.

Plan: Discuss references, the mathematical world, draw some corollaries and questions, introduce understandable equivalent statements for $B\Sigma_2$ and $B\Sigma_3$, and then the proofs.

Does it pay to reflect mathematically?

Thanks to the Erwin Schrödinger International Institute for Mathematics and Physics.

References

These theorems of Jeff's are all in his 1987 PhD. thesis, *Combinatorics in Subsystems of Second Order Arithmetic*. I first learned of them when I worked on the first draft of CJS. Simpson told us about these results and we had Jeff's permission to publish them. They all appeared in the first draft of CJS but some were later removed.

Also see Dzhafarov and Mummert's book, Definition 3.3.1 and Section 6.5, and there is also a earlier paper of Frittaion and Marcone, *Linear extensions of partial orders and reverse mathematics*, see Section 2.

My view here is very influenced by my own 1991 PhD. thesis on c.e. sets.

Mathematical world

$$(\mathbb{N}, \mathcal{P}, +, x, <, 0, 1),$$

where $\mathcal{P} \subseteq 2^{\mathbb{N}}$. We will just call this structure $\mathcal{P}$. Must be at least a model of $PA^- + I\Delta_0 +$ “the exponential function is total” (for coding). Preferred when $\mathbb{N} = \omega$. In that case we want $\mathcal{P}$ closed under Turing reducibility, sometimes closed under paths through trees, the Turing jump, or the ω -jump.

First Order Part of these Principles

$B\Sigma_n$ is equivalent to a Π_{n+2} sentence. $B\Sigma_n$, for $n > 1$, is independent from RCA_0 .

Corollary (Hirst)

- $RCA_0 + RT_2^2$ is not Π_4 -conservative over RCA_0 .
- $RCA_0 + RT_{<\infty}^2$ is not Π_5 -conservative over RCA_0 .

Let's call the question ...

Theorem (CJS)

- $RCA_0 + I\Sigma_2 + RT_2^2 + WKL$ is Π_1^1 -conservative over $RCA_0 + I\Sigma_2$.
- $RCA_0 + I\Sigma_3 + RT_{<\infty}^2 + WKL$ is Π_1^1 -conservative over $RCA_0 + I\Sigma_3$.

Question (CJS)

What is the first order theory of $RCA_0 + RT_2^2$? of $RCA_0 + RT_{<\infty}^2$?

Lots of ongoing work ...

The 6th animal in a zoo

$\mathbb{N} = \omega$ is well ordered and satisfies all instances of induction and hence bounding. In this case REC satisfies $B\Sigma_2$ but not WKL .

Theorem (Harrington)

$RCA_0 + WKL$ is Π_1^1 -conservative over RCA_0 .

Again $B\Sigma_n$, for $n > 1$, is independent from RCA_0 and is equivalent to a Π_{n+2} sentence.

Corollary

$RT_{<\infty}^1$ and WKL are independent over RCA_0 .

Infinite Pigeonhole Principle

If one has infinite many pigeons and puts them into finitely many pigeonholes, at least one pigeonhole will have infinitely many pigeons.

The contrapositive: If one have finite many pigeonholes all containing finitely many pigeons you only have finitely many pigeons (this has been called FUF).

Think of enumeration of a ball x at stage s into a c.e. set W_e as a pigeon being placed into a hole.

The finite union of finite c.e. sets is finite.

Fix n and a computable functions f whose domain includes $[0, n)$.

$$\forall i < n \exists b_i \left[\forall t (W_{f(i), b_i} = W_{f(i), t}) \right] \implies \\ \exists b \forall i < n \left[\forall t (W_{f(i), b} = W_{f(i), t}) \right]$$

The part in the brackets is Π_1 so this an instance of $B\Sigma_2$. $B\Sigma_2$ lets one flip the quantifiers moving the bounded quantifier inward.

In our mathematical world, “ $x \in W_{e,s}$ ” is equivalent to a Δ_1 -formula $\varphi(x, s)$ and every Δ_1 -formula $\varphi(x, s)$ is equivalent to “ $x \in W_{e,s}$ ”, for some e . So these above instances implies $B\Sigma_2$.

Strongly Weihrauch Reducible, $\leq_{sW}$

We will consider the arithmetical statement $\mathcal{A}(I, S)$ which says S is a solution of some instance I of some fixed problem.

Definition

We say $\mathcal{A}_0(I_0, S_0)$ is (*strongly*) *Weihrauch reducible over $\mathcal{P}$* to $\mathcal{A}_1(I_1, S_1)$ iff, there are two Turing functionals Φ and Ψ , such that:

- $\mathcal{P}$ realizes

$$\forall I_0 \forall S_1 (\mathcal{A}_1(\Phi(I_0), S_1)) \Rightarrow \mathcal{A}_0(I_0, \Psi(I_0, S_1)).$$

- If we can do without I_0 , i.e. just use $\Psi(S_1)$ then the reduction is strong.
- Both $\Phi(I_0)$ and $\Psi(I_0, S_1)$ are total functions on all reasonable oracle (sets) from $\mathcal{P}$.

$$RT_{<\infty}^1 <_W RT_2^2$$

Let C_0 be a finite coloring of naturals. Let $\Phi(C_0)(x, y) = 1$ iff $C_0(x) = C_0(y)$. 0 otherwise.

Let H_1 be homogenous for $\Phi(C_0)$. Since there are only finitely many colors H_1 cannot have color 0. So all naturals in H_1 have the same color. Let $\Psi(H_1) = H_1$.

If we want the actual color we need the coloring C_0 . Let $h \in H_1$ and $\Psi(C_0, H_1) = C_0(h)$.

Another option is to use the coloring $\Phi(C_0)(x, y) = C_0(x)$ and work from $SRT_{<\infty}^2$. The coloring is stable, $\lim_y \Phi(C_0)(x, y)$ exists.

$RT_{<\infty}^1$ is computably true but, by Spector, RT_2^2 is not. So $RT_2^2 \not\leq_W RT_{<\infty}^1$.

What about a similar statement for $B\Sigma_3$?

The intersection of finite many cofinite c.e. sets is cofinite.

$$\forall i < n \exists b_i \left[\forall x > b_i \exists s (x \in W_{f(i),s}) \right] \implies \\ \exists b \forall i < n \left[\forall x > b \exists s (x \in W_{f(i),s}) \right]$$

The part in the brackets is Π_2 so this an instance of $B\Sigma_3$. As above these instances implies $B\Sigma_3$.

The contrapositive

The finite intersection of finite many c.e. sets is not cofinite then one of the c.e. sets is not cofinite.

The instance I here is a collection of c.e. sets $\{W_{f(e)} \mid e < n\}$. The solution S is just an e . $\mathcal{A}(I, S)$ here just checks that if the intersection of the sets is not cofinite then $W_{f(e)}$ is not cofinite.

Lets call this principal FICF.

$$FICF <_W SRT^2_{<\infty}$$

Our instance I is a collection of c.e. sets $\{W_{f(e)} \mid e < n\}$ whose intersection is not cofinite. Given a pair x, s . If there is a least y_s and a least e_s such that $x \leq y_s \leq s$ and $y \notin W_{f(e_s),s}$, let $\Phi(I)(x, s) = e_s$. Otherwise let $\Phi(I)(x, y) = n$.

For all x , there is a least $y_x \geq x$ and a least e_x where $y \notin W_{f(e_x)}$. $\lim_s y_s = y_x$ and $\lim_s e_s = e_x$. This coloring is stable and the limit color is not k since the intersection is not cofinite.

Let H_1 be homogenous for $\Phi(I)$ with color $e < n$. $\Psi(H_1) = e$. $W_{f(e)}$ is not cofinite. $\{y_x \mid x \in H_1\} \cap W_{f(e)} = \emptyset$.

What about $B\Sigma_n$, where $n > 3$?

Lemma

- $\{e \mid W_e \text{ is finite}\}$ is Σ_2 -complete.
- $\{e \mid W_e \text{ is cofinite}\}$ is Σ_3 -complete.

Nothing similar for $n > 3$.

THANKS