Free Set, Thin Set and Rainbow Ramsey for the exactly large sets and barriers

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Outline

1. Context and Motivation
2. FS, TS and RRT for exactly large sets
3. FS, TS and RRT for generalized barriers

Context and Motivation

Reverse Mathematics

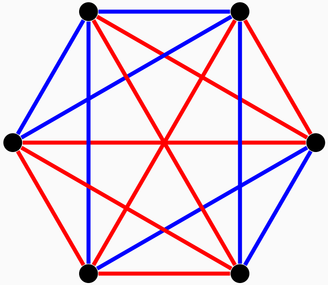
- RCA_0 - Second order arithmetic with comprehension for computable sets.
- ACA_0 - RCA_0 plus arithmetical comprehension.
- $\mathsf{ACA}_0^{'}$ - RCA_0 plus the axiom stating that for every n and for every set X the n-th jump of X exists for all sets X.
- ACA_0^{+} - RCA_0 plus the axiom stating that for every set X the ω-jump of X exists.

Ramsey's theorem

RT^n_k is the statement: "For every coloring $f : [\mathbb{N}]^n \to k$, there is an infinite set $H \subseteq \mathbb{N}$ such that $|f([H]^n)| = 1$".

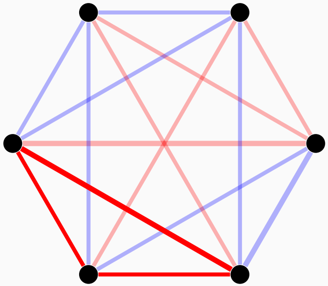
$$\begin{tikzcd}
\forall n \mathsf{RT}^n \arrow[r, leftrightarrow] \arrow[d] & \mathsf{ACA}_0' \arrow[d] \\
\mathsf{RT}^3_2 \arrow[r, leftrightarrow] \arrow[d] & \mathsf{ACA}_0 \arrow[ddd] \\
\mathsf{RT}^2_2 \arrow[d] & \\
\mathsf{RT}^1 \arrow[dr] & \\
 & \mathsf{RCA}_0
\end{tikzcd}$$

Every 2-coloring of the edges of K_6



Motivation - Finite Ramsey Theorem

Every 2-coloring of the edges of K_6, there exists some monochromatic copy of K_3.



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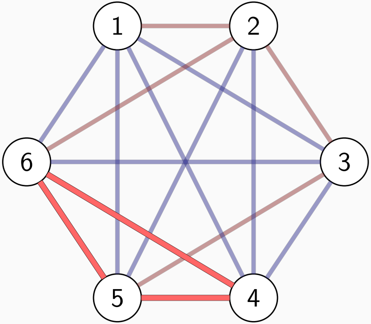
Paris-Harrington (1977) (Strengthened FRT)...not only must the homogeneous subset exist, but its size must also be **larger** than the smallest element in the subset.



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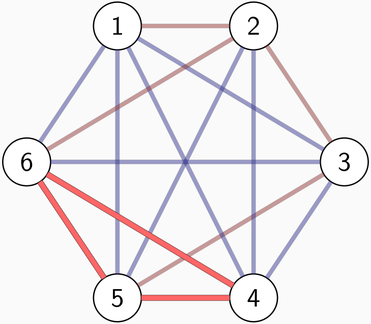

(Exactly) ω-large sets. A finite $s \subseteq \mathbb{N}$ is ω-large if $|s| \geq \min(s) + 1$ and is exactly ω-large if $|s| = \min(s) + 1$.

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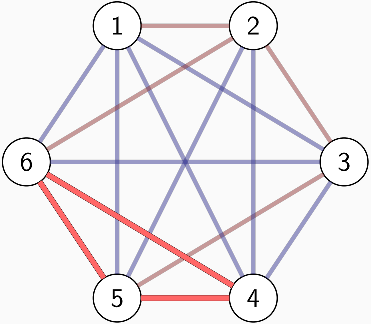

(Exactly) ω-large sets. A finite $s \subseteq \mathbb{N}$ is ω-large if $|s| \geq \min(s) + 1$ and is exactly ω-large if $|s| = \min(s) + 1$. $\{4, 5, 6\}$ is not large. $\{3, 8, 14, 17, 45\}$ is large. $\{3, 8, 14, 17\}$ is ex.large.

Motivation - False Generalizations

Another motivation comes from Ramsey's theorem for colorings of finite subsets of $\mathbb{N}$.

Definition

Let $\mathsf{RT}^{<\omega}$ be the statement that for every coloring of $f : [\mathbb{N}]^{<\omega} \to 2$ there exists an infinite set H such that $|f([H]^{<\omega})| = 1$.

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Ramsey's theorem of exactly large sets is one way out:
(We write $[H]^{!\omega}$ for the set of all the exactly ω-large subsets of H.)

Background- Large Ramsey Theorem

Large Ramsey's Theorem(Pudlák-Rödl (1982) & **Farmaki (2002))**
($\mathsf{RT}_2^{!\omega}$) For every coloring $f : [\mathbb{N}]^{!\omega} \to 2$, there exists an infinite set H such that f is constant on $[H]^{!\omega}$.

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Theorem (Carlucci-Zdanowski (2012)

Over RCA_0, $RT^{!\omega}$ is equivalent to ACA_0^+.

$$\begin{tikzcd}
\mathrm{ACA}_0^+ \arrow[r, leftrightarrow] \arrow[d] & \mathrm{RT}^{!\omega} \arrow[d] \\
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Question

What about consequences of Ramsey's Theorem?

FS, TS and RRT for exactly large sets

Free, Thin and Rainbow Theorems

Free Set Theorem(Friedman-FOM - 1999)

(FS^n): If $f : [\mathbb{N}]^n \to \mathbb{N}$, then there exists an infinite $H \subseteq \mathbb{N}$ such that $s \in [H]^n$ and $f(s) \in H$ imply $f(s) \in s$.

Thin Set Theorem (Friedman-FOM - 1999)

(TS^n): If $f : [\mathbb{N}]^n \to \mathbb{N}$, then there exists an infinite set $H \subseteq \mathbb{N}$ such that $f([H]^n) \neq \mathbb{N}$.

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Rainbow Ramsey Theorem

(RRT^n_k): Let $n, k > 0$. For all $f : [\mathbb{N}]^n \to \mathbb{N}$ if $|f^{-1}(z) \leq k|$ for all $z \in \mathbb{N}$, then there exists an infinite $H \subseteq \mathbb{N}$ such that f is injective on $[H]^n$.

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Similarly, the generalizations $\mathrm{FS}^{<\omega}, \mathrm{TS}^{<\omega}, \mathrm{RRT}^{<\omega}_2$ fail.
(For $TS^{<\omega}$, take $f(s) = |s|$.)

Analysis of free and thin set

FS and TS were studied by Cholak, Giusto, Hirst and Jockusch (2001).

- $\mathsf{RCA}_0 \vdash \mathsf{FS}^1$ but not FS^2.
- $\mathsf{RCA}_0 \vdash \mathsf{FS}^{\mathsf{n}+1} \rightarrow \mathsf{FS}^{\mathsf{n}}$ respectively $TS^{n+1} \rightarrow TS^{n}$.
- $\mathsf{RCA}_0 \vdash \forall \mathsf{nFS} \rightarrow \forall \mathsf{nTS}^{\mathsf{n}}$.

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- $\mathsf{RCA}_0 \vdash \forall n \mathsf{FS} \to \forall n \mathsf{TS}^n$.
- $\mathsf{RCA}_0 \vdash \mathsf{RT}^n_2 \to \mathsf{FS}^n \to \mathsf{TS}^n$.

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Wang(2014) showed that $\mathsf{RCA}_0 \vdash \mathsf{FS}^{\mathsf{n}} \to \mathsf{RRT}^{\mathsf{n}}_2$.

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Wang(2014) showed that $\mathsf{RCA}_0 \vdash \mathsf{FS}^{\mathsf{n}} \to \mathsf{RRT}^{\mathsf{n}}_2$.

They have diagonalizing power comparable to Ramsey's Theorem (Cholak et al. 2001, Csima and Mileti 2009):

$$\Sigma^0_{\mathsf{n}} < \mathsf{FS}^{\mathsf{n}}, \mathsf{TS}^{\mathsf{n}}, \mathsf{RRT}^{\mathsf{n}}_2 \leq \Pi^0_{\mathsf{n}}.$$

Free, Thin, Rainbow are weak

These principles have no coding power.

Wang (2014) $\mathsf{FS}, \mathsf{TS}, \mathsf{RRT}$ do not imply ACA_0/code the jump.

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$$\begin{tikzcd}
 & RRT^{2n+1}_2 \arrow[d, "\text{Patey}"] & RRT^{n+1}_2 \arrow[d, "\text{Patey}"] \arrow[dd, bend left, "\text{Csima,Mileti}"] & & RT^2_2 \arrow[d, "\text{Cholak etal}"'] \arrow[dd, bend right, "\text{Hirst}"'] \\
RT^n_2 \arrow[r, "\text{Cholak etal}"] \arrow[drr, "\text{Galvin}"] & FS^n \arrow[r, "\text{Cholak etal}"] \arrow[dr, "\text{Wang}"] & TS^n \arrow[d, "\text{Patey}"] & & FS^2 \arrow[d, "?"] \\
 & & RRT^n_2 \arrow[r, "\text{Csima,Mileti}"] & RRT^2_2 \arrow[r, "\text{Slaman}", "/" marking] & B\Sigma^0_2
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\end{tikzcd}
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Observation: $\mathsf{TS}^{<\omega} \leq_{\mathrm{sW}} \mathsf{FS}^{<\omega}$.

Question: What about generalizations to exactly large sets?

Large Free Set

Large Free Set Theorem($\mathsf{FS}^{!\omega}$)

For every coloring $f : [\mathbb{N}]^{!\omega} \to \mathbb{N}$ there exists an infinite set H free for f.

Define $\mathsf{TS}^{!\omega}, \mathsf{RRT}^{!\omega}$ similarly.

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Generalization Results:

$\mathsf{FS}^{!\omega}$ implies $\mathsf{TS}^{!\omega}$ and $\mathsf{RRT}^{!\omega}_2$ over RCA_0 (proofs of Cholak et al and Wang translate verbatim).

$\mathsf{RT}^{!\omega} \to \mathsf{RRT}^{!\omega}_k$ over RCA_0 (Galvin's argument translates verbatim).

What about $\mathsf{RT}^{!\omega}_2 \to \mathsf{FS}^{!\omega}$?

Large Free Set

Theorem (Carlucci, G., Le Houérou, Levy Patey)

$\mathsf{FS}^{!\omega} \leq_{\mathsf{sW}} \mathsf{RT}^{!\omega}_2$.

Let $f : [\mathbb{N}]^{!\omega} \to \mathbb{N}$ be an instance of $FS^{!\omega}$.
Define $g : [\mathbb{N}]^{!\omega} \to 2$ by induction.
For $H = \{s_0, s_1, \ldots\}$ let $H' = \{s_0 - 1, s_1 - 1, \ldots\}$.
A concrete example:

s	3	7	11	14
s'	2	6	10	13

$$g(s) = \begin{cases} 0 & \text{if } f(s') \in \{2, 6, 10\} \\ 1 & \text{if } f(s') > 14, \\ 0 & \text{if } f(s') \in (10, 13), \\ 1 - g(2, 7, 11) & \text{if } f(s') < 2, \text{(say } f(s') = 1) \\ 1 - g(3, 5, 11, 14) & \text{if } f(s') \in (2, 6) \text{ or } (6, 10), \text{(say } f(s') = 4) \end{cases}$$

By $\mathsf{RT}^{!\omega}_2$ let $H \subseteq \mathbb{N}$ be an infinite set such that g is constant on $[H]^{!\omega}$.

Claim $H' = \{s_0 - 1, s_1 - 1, \ldots,\}$ is f-free.

Let $\mathrm{H} = \{3, 7, 11, 14, 22, 31...\}$, then $\mathrm{H}' = \{2, 6, 10, 13, \ldots\}$.

Pick $s' = \{2, 6, 10\} \in H'$, then $\{3, 7, 11\} \in H$.

Pick $s_3 \in H$ to be the the next element of $s = \{3, 7, 11, 14\}$.

- H is homogeneous for color 0.

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Large Free Set Theorem

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- H is homogeneous for color 1.

$$g(s) = \begin{cases} 0 & \text{if } f(s') \in \{2, 6, 10\} \\ \mathbf{1} & \text{if } f(s') > 14, \\ 0 & \text{if } f(s') \in (10, 13), \\ \textbf{1-g(2, 7,11)} & \text{if } f(s') < 2, \text{(say } f(s') = 1) \\ \textbf{1-g(3, 5, 11,14)} & \text{if } f(s') \in (2, 6) \text{ or } (6, 10), \text{(say } f(s') = 4) \end{cases}$$

Lower Bounds

Theorem (Carlucci, G., Le Houérou, Levy Patey)

$\mathsf{TS}^{!\omega}$ *and* $\mathsf{FS}^{!\omega}$ *proves the existence of* $\emptyset^{(\omega)}$*, the* ω*-th jump and imply* ACA_0^+*.*

Lower Bounds

Ideas from the proof.

- Sufficient to show the statement for $TS^{!\omega}$ as any free set will be thin.
- (Dorais et al) For $n \in \mathbb{N}$, there exists a computable coloring $f : [\mathbb{N}]^{n+2} \to 2^n$ such that every infinite f-thin set computes $\emptyset^{'}$.
- For every $n \geq 2$, we can get a computable coloring $f_n : [\mathbb{N}]^n \to n$ such that every infinite f_n-thin set uniformly computes $\emptyset^{'}$.
- Combining these instances, we can compute $\emptyset^{(\omega)}$ by tranforming the $\emptyset^k$-computable colorings into a computable coloring $f : [\mathbb{N}]^{n+k} \to \mathbb{N}$.
- We obtain a coloring $f : [\mathbb{N}]^{!\omega} \to \mathbb{N}$ such that $f(s_0, \ldots, s_{s_0}) = f_{s_0}(s_1, \ldots, s_{s_0})$. An f-thin set $H = \{s_0, \ldots\}$ will be f_{s_i}-thin for every i thus, it computes $\emptyset^{'}$. Using the previous point, $H \geq_T \emptyset^{(\omega)}$.

Lower bounds

There is a computable instance of $\mathsf{RRT}^{!\omega}_2$ with no arithmetical solution (using the uniformity of Csima-Mileti result $\mathsf{RRT}^{\mathsf{n}+1}_2 \to \mathsf{RRT}^{\mathsf{n}}_2$).

$\mathsf{TS}^{!\omega} \leq_{\mathsf{sW}} \mathsf{RRT}^{!(\omega+1)}_2$ (translation from Patey's result $\mathsf{RRT}^{\mathsf{n}+1}_2 \to \mathsf{TS}^{\mathsf{n}}$).

(Corollary) There exists a computable instance of $\mathsf{RRT}^{!(\omega+1)}_2$ such that every solution computes $\emptyset^{(\omega)}$.

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Theorem (Carlucci, G., Le Houérou, Levy Patey)

$\mathsf{RRT}^{!\omega}$ *has the strong cone avoidance property.*

FS, TS and RRT for generalized barriers

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- $\mathsf{RT}^{!\omega}$ is also the base case of a generalization of Ramsey's Theorem due to Nash-Williams to colorings of barriers.

Theorem (Barrier Ramsey Theorem , RT^{B})

Let $B \subseteq [\mathbb{N}]^{<\omega}$. For every finite coloring $f : B \to k$ there exists $H \in [\mathbb{N}]^{\omega}$ such that the restriction $B|H$ is homogeneous for the coloring.

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Free, Thin, Rainbow for barriers

Definition (Free Set Theorem for barriers)

Let $B \subseteq [\mathbb{N}]^{<\omega}$. For every $f : B \to \mathbb{N}$ then there exists $H \in [\mathbb{N}]^{\omega}$ such that $B|H$ is free for f.

TS^{B} and $\mathsf{RRT}^{\mathsf{B}}$ can be defined similarly.

Free set for barriers

(Picking up from Cholak, Giusto, Hirst, Jockusch proof)

- Case ($f(s) < min(s)$) adapts.

The following property of exactly large sets generalizes to barriers.
If $s \in [X]^{!\omega}$, $n \in X$ and $n < min(s)$ then $\{n\} \cup \{s_0, s_1, \ldots, s_{n-1}\} \in [X]^{!\omega}$.

Lemma

Let B be a barrier on X. Let $s \in B$ and $n \in X$ such that $n < min(s)$. Then there exists an $s' \sqsubset s$ such that $\{n\} \cup s' \in B$.

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Lemma

Let B be a barrier on X. Let $s \in B$ and $n \in X$ such that $n < min(s)$. Then there exists an $s' \sqsubset s$ such that $\{n\} \cup s' \in B$.

- Case ($s_i < f(s) < s_{i+1}$): prove directly by transfinite induction.

Idea: Identify $f : B \to \mathbb{N}$ with the union of functions $f_{n_k} : B_{n_k} \to \mathbb{N}$ where each B_{n_k} is a "thinning" of B and $f_n(s) = f(\{n_k\} \cup s)$ for $s \in B_{n_k}$.
Find an infinite sequence $S = (n_k)_{k \in \mathbb{N}}$ such that $B|S$ is free for f.

Diagonalization for general barriers

Clote 1984/1987 analyzed Ramsey's Theorem for barriers relative to the transfinite jump hierarchy.

Definition (Canonical barriers B_a for $a \in \mathcal{O}$)

1. $B_1 = \{\}$
2. $B_2 = \{\{n\} : n \in \omega\}$
3. $B_{2^a} = B_2 * B_a = \{\{n\} \cup t\}$ for $n \in \omega$ and $t \in B_a$ st $n < min(t)$.
4. $B_{3 \cdot 5^z} = \bigcup_{n \in \mathbb{N}} \{n\} * B_{\{z\}(n)} * B_{\{z\}(n-1)} * \cdots * B_{\{z\}(0)}$.

- Assous proved that each B_a is a barrier with base $\mathbb{N}$ and order type $ot(B_a) = \omega^{|a|}$.

- Limit is taken over the canonical barriers $B_a,\ a \in \mathcal{O}$.

- Idea here is that canonical barriers of order type $\omega^{|a)}$ in order to computably approximate $\emptyset^{\alpha}$.

Diagonalization for general barriers

Theorem (Generalized Limit Lemma (Clote, 1984))

There exists a partial computable function g such that for all ordinal notations $a \in \mathcal{O}$, for all $e, x \in \mathbb{N}$ and $s \in \mathbb{N}^{<\omega}$, $g(a, e, x, s) \downarrow$ and

$$\varphi_e^{\emptyset^{(a)}}(x) = y \Longrightarrow \lim_{s \in B(a)} g(a, e, x, s) = y.$$

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Theorem (Anti-basis theorem for RT^B (Carlucci, G.))

For each $a \in \mathcal{O}$ there is a computable coloring of the barrier $B_2 * B_a$ with no $\emptyset^{(a)}$-computable thin set.

Proof idea: Jockusch's Δ^0_2-omission for RT^2_2 + Clote's Limit Lemma.

Bibliography

1. P. A. Cholak and M. Giusto and J. L. Hirst and C. G. Jockusch. *Free sets and Reverse Mathematics.*, In: S. G. Simpson (ed.) *Reverse Mathematics 2001*. Lecture Notes in Logic, Cambridge Univ. Press, 104-119, 2005.
2. P. A. Cholak and L. Patey, *Thin set theorems and cone avoidance* Transactions of the AMS, 373:4, 2743–2773, 2020.
3. L. Carlucci and K. Zdanowski. The strength of Ramsey's theorem for coloring relatively large sets. J. Symb. Log., 79(1):89–102, 2014.
4. B. F. Csima and J. R. Mileti, *The Strength of the Rainbow Ramsey Theorem.* The Journal of Symbolic Logic,74:4, 1310–1324, 2009.
5. W. Wang. *Some logically weak Ramseyan theorems.* Adv. Math., 261:1–25, 2014.

Thank you!