The strength of Ramsey's theorem over a weaker base theory

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Theorem (McAloon; Friedman, Simpson)

$\mathsf{RCA}_0 \vdash \mathsf{RT} \Leftrightarrow \mathsf{ACA}_0'$.

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In 2018, Slaman and Yokoyama proved that $\mathsf{RCA}_0 + \mathsf{RT}^2_{<\infty}$ is Π^1_1-conservative over $\mathsf{B}\Sigma^0_3$.

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- $\mathsf{RCA}_0 + \mathsf{RT}^2_2$ is $\forall\Pi^0_4$-conservative over $\mathsf{RCA}_0 + \mathsf{B}\Sigma^0_2$
  (Le Houérou, Patey, Yokoyama, 2024).

A weaker base theory

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- To strengthen some reversals known to hold over RCA_0.

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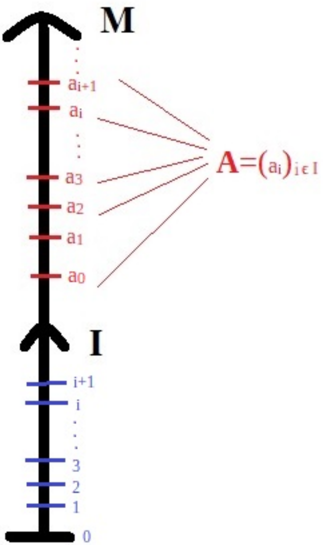

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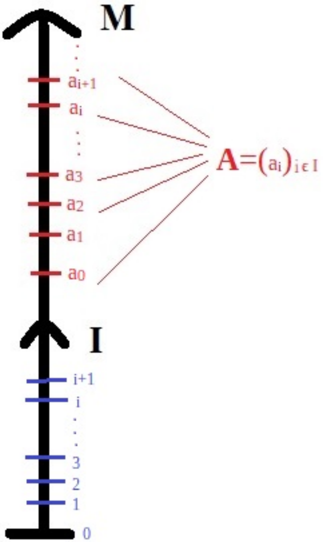

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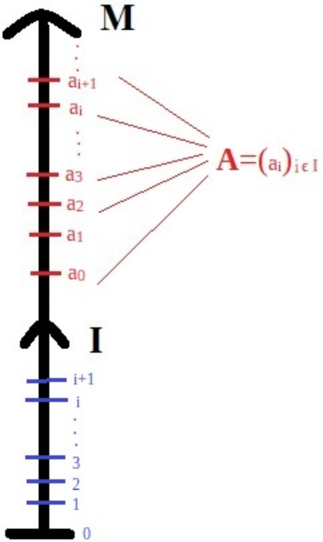

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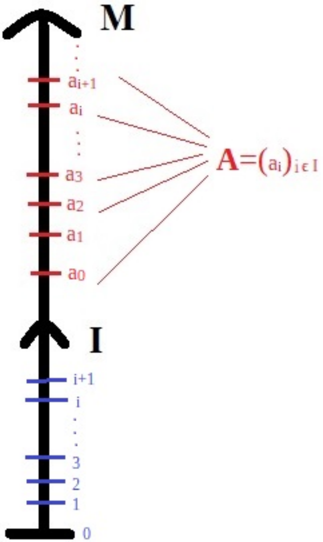

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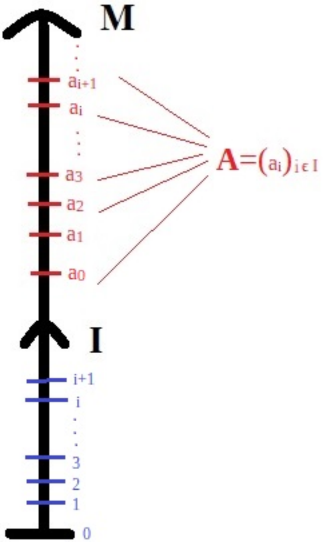

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$\mathrm{SSy}(M) := \mathrm{Cod}(M/\omega)$

For all $n, k \geq 2$, RCA_0^* proves $\mathsf{RT}^n_k \Rightarrow \mathsf{RT}^n_{k+1}$.

Thus, from now on, we consider only RT^n_2 for $n \geq 2$.

Equivalence with relativization to a Σ_1^0-definable cut

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Theorem

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Corollary

Let $(M, \mathcal{X}) \models \mathsf{RCA}_0^* + \neg \mathsf{I}\Sigma_1$. *Then*

$$(M, \mathcal{X}) \models \mathsf{RT}^n_2 \quad \textit{iff} \quad (M, \Delta_1\text{-}\mathrm{Def}(M)) \models \mathsf{RT}^n_2.$$

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Chong-Mourad coding lemma

Let $(M, \mathcal{X}) \models \mathsf{RCA}_0^*$ *and* I *be a cut in* M*. If both* $Y \subseteq I$ *and* $I \setminus Y$ *are* Σ^0_1*-definable in* M*, then* $Y \in \mathrm{Cod}(M/I)$*.*

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$\mathsf{RCA}_0^* + \mathsf{RT}_2^n$ proves the following Π_4 sentence:

$$\neg \mathrm{I}\Sigma_1 \Rightarrow \Delta_1\text{-}\mathsf{RT}_2^n,$$

where Δ_ℓ-RT_2^n says: *For every* Δ_ℓ*-definable 2-colouring of* $[\mathbb{N}]^n$ *there is a* Δ_ℓ*-definable infinite homogeneous set.*

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2. *the theory* $\mathsf{B}\Sigma_\ell + \exp + \Delta_\ell$-$\mathsf{RT}^n_2$ *is consistent.*

To prove (1), formalize the argument of (Jockusch 1972).

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To prove (1), formalize the argument of (Jockusch 1972).
For (2), consider a model $M \vDash \mathsf{B}\Sigma_\ell + \neg\mathsf{I}\Sigma_\ell$ with Σ_ℓ-definible ω and $\mathrm{SSy}(M) \vDash \mathsf{RT}_2^n$. Apply the 'cut equivalence theorem' to the structure $(M, \Delta_\ell\text{-}\mathrm{Def}(M))$.

Provability of computational bounds on RT^n_2

Lemma

Let $n \geq 3$ *and* $(M, \mathcal{X}) \models \mathsf{RCA}_0^* + \mathrm{RT}^n_2$*. If* $M \models \mathrm{I}\Sigma_\ell$ *then* $0^{(\ell)} \in \mathcal{X}$*. As a consequence,* $\Delta_{\ell+1}\text{-}\mathrm{Def}(M) \subseteq \mathcal{X}$ *and* $M \models \mathrm{B}\Sigma_{\ell+1}$*.*

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Lemma

Let $n \geq 3$ and $(M, \mathcal{X}) \models \mathrm{RCA}^*_0 + \mathrm{RT}^n_2$. If $M \models \mathrm{I}\Sigma_\ell$ then $0^{(\ell)} \in \mathcal{X}$. As a consequence, $\Delta_{\ell+1}\text{-}\mathrm{Def}(M) \subseteq \mathcal{X}$ and $M \models \mathrm{B}\Sigma_{\ell+1}$.

For $\ell = 1$ take Jockusch's computable colouring of triples whose solutions compute $0'$: for $x < y < z$ let $c(x, y, z) = 0$ if there is a Turing machine with a code below x that halts below z but not below y. By RT^3_2, there is a homogeneous set H, and by $\mathrm{I}\Sigma_1$ it must have colour 1. To check whether $e \in 0'$ take any $x, y \in H$ such that $e < x < y$ and execute the computation Φ_e on e until the y-th step.
For the other cases proceed by induction up to ℓ and relativize the case $\ell = 1$.

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- The Π_3-part of R^2 is $\mathsf{B}\Sigma_1 + \exp$.
- The Π_4-part of R^2 is strictly weaker then $\mathsf{B}\Sigma_2$ but does not follow from $\mathsf{I}\Sigma_1$:
- $\mathsf{RCA}^*_0 + \mathsf{RT}^2_2 \vdash \mathsf{C}\Sigma_2$ but $\mathsf{I}\Sigma_1 \nvdash \mathsf{C}\Sigma_2$,
  where $\mathsf{C}\Sigma_2$ is a Π_4-sentence saying 'there is no Σ_2-definable injection $f : \mathbb{N} \to \mathbb{N}$ with a bounded range'.

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An $\mathcal{L}_2$ sentence σ is called pSO if there exists a sentence γ of second-order logic in a relational language $(\leq, R_1, \ldots, R_k)$ such that σ expresses:

> *For every relations $R_1, \ldots, R_k$ on $\mathbb{N}$ and every infinite set $D \subseteq \mathbb{N}$, there exists an infinite set $H \subseteq D$ such that $(H, \leq, R_1, \ldots, R_k) \models \gamma$.*

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The 'cut equivalence theorem' holds for any pSO sentence σ:

$$(M, \mathcal{X}) \vDash \sigma \quad \text{iff} \quad (I, \mathrm{Cod}(M/I)) \vDash \sigma,$$

where $(M, \mathcal{X}) \vDash \mathsf{RCA}_0^*$ and I is a Σ_1^0-definable proper cut.

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Theorem

Let σ be a pSO sentence. Then the following are equivalent:

(i) $\mathsf{RCA}_0^* + \sigma$ *is* Π^1_1*-conservative over* RCA_0^*,

(ii) $\mathsf{RCA}_0^* + \neg \mathsf{I}\Sigma^0_1 \vdash \sigma$,

(iii) $\mathsf{WKL}_0^* \vdash \sigma$.

Moreover, if $\mathsf{WKL}_0 \nvdash \sigma$, *then* $\mathsf{RCA}_0^* + \sigma$ *is not arithmetically conservative over* RCA_0^*.

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Theorem

Let σ be a pSO sentence such that there exists an ω-model of the theory $\mathsf{WKL}_0 + \sigma$. *Then* $\mathsf{WKL}_0^* + \sigma$ *is* $\forall\Pi^0_3$*-conservative over* RCA_0^*.

Other Ramsey-like principles

CAC = For every partial order $(\mathbb{N}, \preceq)$ there exists an infinite set $S \subseteq \mathbb{N}$ which is a $\preceq$-chain or $\preceq$-antichain.

ADS = For every linear order $(\mathbb{N}, \preceq)$ there exists an infinite set $S \subseteq \mathbb{N}$ which is an $\preceq$-ascending or $\preceq$-descending sequence.

CRT^2_2 = for every $c \colon [\mathbb{N}]^2 \to 2$ there exists an infinite $S \subseteq \mathbb{N}$ such that $c \restriction S$ is stable, i.e. for every $x \in S$ there exists $y \in S$ such that for all $z \in S$ if $z \geq y$, then $c(x, y) = c(x, z)$.

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- Each of $\mathsf{CAC}, \mathsf{ADS}$ and CRT^2_2 holds in some model of RCA_0^* of the form $(M, \Delta_1\text{-}\mathrm{Def}(M))$. That is, in the absence of $\mathsf{I}\Sigma^0_1$ they may all be computably true.

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COH: For each sequence $(R_n)_{n\in\mathbb{N}}$ of subsets of $\mathbb{N}$, there exists an unbounded set C which is cohesive for $(R_n)_{n\in\mathbb{N}}$ (i.e. for every $i \in \mathbb{N}$ either $C \subseteq^* R_i$ or $C \subseteq^* \overline{R}_i$).

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Theorem (Mengzhou Sun, 2025)

$\mathsf{RCA}^*_0 + \mathsf{COH}$ *implies* $\mathsf{I}\Sigma^0_1$.

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RT^2_2 is $\forall\Pi^0_3$-conservative over both RCA_0 and RCA_0^*, but the proofs are quite different. Can anything more be said about this?

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I.e., for every elementary computable function f there exists a Σ_1 sentence σ and its proof δ in $\mathsf{RCA}^*_0 + \mathsf{RT}^2_2$ such that every proof of σ in RCA^*_0 has size greater than $f(|\delta|)$.

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The proof of speedup for $\mathrm{RCA}_0^* + \mathrm{RT}_2^2$ relies on the exponential lower bound on Ramsey numbers for finite version of RT_2^2:

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- Does RT^3_2 imply RT^4_2 over RCA^*_0?
- Does ADS or CAC imply CRT^2_2 over RCA^*_0?
- Does $\mathsf{RCA}^*_0 + \mathsf{RT}^2_2$ imply $\mathrm{I}\Sigma_1 \Rightarrow \mathrm{B}\Sigma_2$?
- Is there a *nice* axiomatization of R^2?
- Is $\mathsf{RCA}^*_0 + \mathsf{CRT}^2_2$ polynomially simulated by RCA^*_0 with respect to $\forall\Pi^0_3$ sentences?

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Thank you!