Progetto PRIN 2022 - Models, sets and classifications

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RS Dimonte

Some results about the reverse mathematics of dimension of posets

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Joint work with Andrea Volpi (originally also with Marta Fiori Carones)

August 9, 2025

Combinatorics (and much more)
in subsystems of second-order arithmetic:
a conference in honor of Jeff Hirst

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UNIVERSITÀ DEGLI STUDI DI UDINE HIC SUNT FUTURA

Outline

1. **Dimension of Posets**
2. **Reverse mathematics of** DBi_n **and** DBc_n
3. **Reverse mathematics of** DB_p

1. Dimension of Posets
2. Reverse mathematics of $\mathsf{DBi_n}$ and $\mathsf{DBc_n}$
3. Reverse mathematics of $\mathsf{DB_p}$

Posets

Let $(P, \preceq)$ be a poset. By $|$ we denote the incomparability relation: $x \mid y$ if and only if $x \not\preceq y$ and $y \not\preceq x$.

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A linear order, or chain, is a poset in which any two distinct elements are comparable.

A poset $(P, \preceq_1)$ extends a poset $(P, \preceq_2)$ if $\preceq_2 \subseteq \preceq_1$.

A linearization of $(P, \preceq)$ is an extension of $(P, \preceq)$ to a linear order.

Dimension

Definition

Let $\mathcal{L}$ be a family of linearizations of $(P, \preceq)$.
$\mathcal{L}$ realizes $(P, \preceq)$ if $\bigcap \mathcal{L} = \preceq$.

In practice, $\mathcal{L}$ realizes $(P, \preceq)$ means that for every $x, y \in P$ with $x \mid y$ there exist $\trianglelefteq_1, \trianglelefteq_2 \in \mathcal{L}$ such that $x \trianglelefteq_1 y$ and $y \trianglelefteq_2 x$.

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Definition

The dimension $\dim(P, \preceq)$ of $(P, \preceq)$ is the least cardinality of a realization of $(P, \preceq)$.

The dimension of a chain is 1, the dimension of an antichain with at least two elements is 2.

$$\dim(P, \preceq) \leq \max\{1, |\{\, (x, y) \in P^2 : x \mid y \,\}|\}$$

but this bound is usually very rough, because a single linearization can take care of many incomparabilities.

Dimension always exists in RCA_0

Proposition

RCA_0 *proves that every (countable) poset has a linearization, and this can be proved uniformly: if we have a (countable) sequence of linear orders there is a sequence of linerarizations.*

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Proposition

RCA_0 *proves that for each (countable)* $(P, \preceq)$ *there exists a set* $\{\trianglelefteq_n : n \in \mathbb{N}\}$ *of linearizations that realizes* $(P, \preceq)$, *i.e. such that for each* $x, y \in P$ $x \preceq y$ *if and only if* $x \trianglelefteq_n y$ *for every* $n \in \mathbb{N}$.

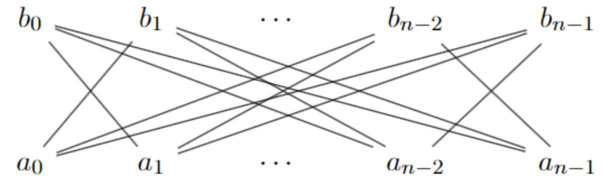
Thus in RCA_0 every poset has a (countable) dimension.

The poset F_n

The basic example of a poset of dimension n:

Definition

Let $n > 1$ and $F_n = \{a_i, b_i : i < n\}$. We equip F_n with the relation $a_i \prec b_j$ whenever $i \neq j$.



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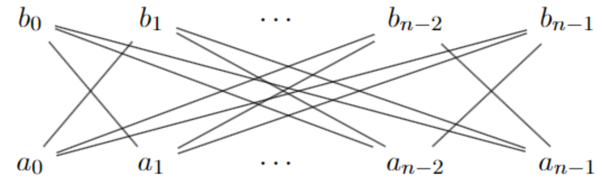

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Lemma

RCA_0 *proves that if* $\{\trianglelefteq_0, \ldots, \trianglelefteq_{n-1}\}$ *realizes* F_n, *then for each* $i < n$ *there is exactly one* $k < n$ *such that* $b_i \triangleleft_k a_i$.

Bounding Theorems

Theorem (Hiraguti, 1955)

Let $(P, \preceq)$ *be a poset:*

- *if* $x_0 \in P$ *then* $\dim(P, \preceq) \leq \dim(P \setminus \{x_0\}, \preceq) + 1$*;*
- *if* $C \subseteq P$ *is a chain then* $\dim(P, \preceq) \leq \dim(P \setminus C, \preceq) + 2$*;*
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- $\mathsf{DB_p}$: if $x_0 \in P$ then $\dim(P, \preceq) \leq \dim(P \setminus \{x_0\}, \preceq) + 1$.
- $\mathsf{DBi_n}$: if $n > 0$ and $C_0, \ldots, C_{n-1} \subseteq P$ are n pairwise incomparable chains, then $\dim(P, \preceq) \leq \dim(P \setminus \bigcup_{i<n} C_i, \preceq) + \max\{2, n\}$.

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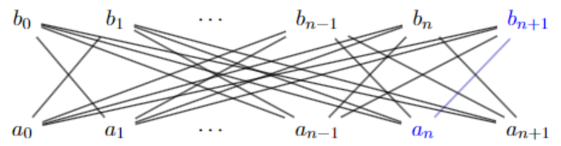
- $\mathsf{DBc_n}$: if $n > 0$ and $C_0, \ldots, C_{n-1} \subseteq P$ are n chains, then $\dim(P, \preceq) \leq \dim(P \setminus \bigcup_{i<n} C_i, \preceq) + 2n$.

$\mathsf{DBi}_1 = \mathsf{DBc}_1$ is optimal

Example

Consider F_{n+2} and the chain $C = \{a_n, b_{n+1}\}$. The poset $F_{n+2} \setminus C$ is a copy of F_{n+1} with an extra comparability.

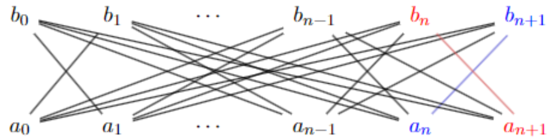


$\dim(F_{n+2} \setminus C) = n$ and $\dim(F_{n+2}) = \dim(F_{n+2} \setminus C) + 2$.

DBi_2 is optimal

Example

Consider F_{n+2}, $C_0 = \{a_{n+1}, b_n\}$ and $C_1 = \{a_n, b_{n+1}\}$. The poset $F_{n+2} \setminus C_0 \cup C_1$ is a copy of F_n.



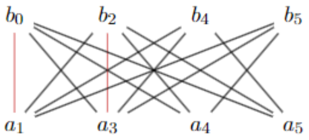
$\dim(F_{n+2} \setminus C_0 \cup C_1) = n$ and
$\dim(F_{n+2}) = \dim(F_{n+2} \setminus C_0 \cup C_1) + 2.$

DBc_2 **is optimal**

Do the chains in DBi_2 really have to be incomparable?

Example

Consider F_6, $C_0 = \{a_0, b_1\}$ and $C_1 = \{a_2, b_3\}$. The poset $F_6 \setminus C_0 \cup C_1$ is a copy of F_4 with two extra comparabilities.



$\dim(F_6 \setminus C_0 \cup C_1) = 2$ and $\dim(F_6) = \dim(F_6 \setminus C_0 \cup C_1) + 4$.

Reverse mathematics of $\mathsf{DBi_n}$ and $\mathsf{DBc_n}$

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Two lemmas about linearizations

Lemma (Cholak-M-Solomon 2004)

Within RCA_0, *the following are equivalent:*

1. WKL_0;
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2. *let* $(P, \preceq)$ *be a poset and* $C_0, C_1 \subseteq P$ *incomparable chains; there exists a linearization* $\trianglelefteq$ *such that for* $c_0 \in C_0$*,* $c_1 \in C_1$ *and* $x \in P$*,* $c_0 \mid x$ *implies* $x \trianglelefteq c_0$ *and* $c_1 \mid x$ *implies* $c_1 \trianglelefteq x$*.*
   [$\trianglelefteq$ *puts* C_0 *at the top and* C_1 *at the bottom of* P*]*

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Both lemmas are uniform: WKL_0 proves the version for countable sequences.

WKL_0 **proves** DBi_n **and** DBc_n

To prove DBi_2 in WKL_0 let $C_0, C_1 \subseteq P$ be incomparable chains, and $\{\trianglelefteq_0, \dots, \trianglelefteq_{m-1}\}$ realize $(P \setminus C_0 \cup C_1, \preceq)$.

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Therefore $\dim(P, \preceq) \leq m + 2$.

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$\mathsf{DBi_1}$ follows from $\mathsf{DBi_2}$ (take one of the chains to be empty).

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To prove $\mathsf{DBc_n}$ we apply the lemma to the pairs $C_i, \emptyset$ and $\emptyset, C_i$ for every new chain C_i.

DBi_1 **implies** WKL_0**, step 1**

Let f, g be injective functions with disjoint images.
We need $(P, \preceq)$ so that every family of linearizations realizing $(P, \preceq)$ computes a set separating $\mathrm{ran}(f)$ from $\mathrm{ran}(g)$.

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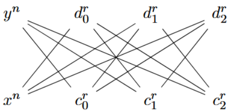
$P = \{\, x^n, y^n : n \in \mathbb{N} \,\} \cup \{\, c^r_j, d^r_j, p^r_j, q^r_j : j < 3, r \in \mathbb{N} \,\}$ consists of layers ordered like ω:

- if n does not belong to the ranges of f and g then layer n consists of the antichain $\{x^n, y^n\}$;
- if $n = f(r)$ then layer n is a copy of F_4:



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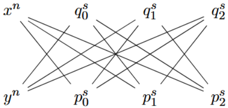

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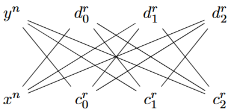

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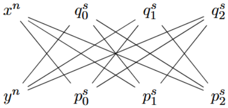
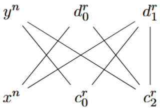
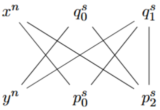


Let $C = \{c^r_1, d^r_2, p^s_1, q^s_2 : r, s \in \mathbb{N}\}$, which is a chain.

DBi_1 **implies** WKL_0, **step 2**

In $(P \setminus C, \preceq)$ each layer is either an antichain of size two or





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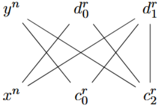

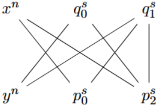

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Distinct layers are linearly ordered, so that $\dim(P \setminus C, \preceq) = 2$ and two computable linearizations that realize it.

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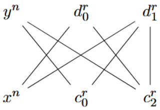

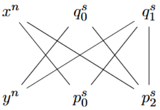

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Distinct layers are linearly ordered, so that $\dim(P \setminus C, \preceq) = 2$ and two computable linearizations that realize it.

By DBi_1, $\dim(P, \preceq) \leq 4$. Let $\{\trianglelefteq_0, \trianglelefteq_1, \trianglelefteq_2, \trianglelefteq_3\}$ realize $(P, \preceq)$.

If $n \in \operatorname{ran}(f)$ then $|\{\, i < 4 : y^n \trianglelefteq_i x^n \,\}| = 1$, while if $n \in \operatorname{ran}(g)$ then $|\{\, i < 4 : y^n \trianglelefteq_i x^n \,\}| = 3$.

$\{\, n : |\{\, i < 4 : y^n \trianglelefteq_i x^n \,\}| = 1 \,\}$ separates $\operatorname{ran}(f)$ from $\operatorname{ran}(g)$.

The reverse mathematics of bounding theorems for chains

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Let $n > 0$*. Within* RCA_0*, the following are equivalent:*

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The reversals from $\mathsf{DBi_n}$ when $n \geq 4$ require to have the poset not completely layered.

1. Dimension of Posets
2. Reverse mathematics of DBi_n and DBc_n
3. **Reverse mathematics of DB_p**

Proving $\mathsf{DB_p}$: $\dim(P, \preceq) \leq \dim(P \setminus \{x_0\}, \preceq) + 1$

Let $\{\trianglelefteq_0, \ldots, \trianglelefteq_{m-1}\}$ realize $(P \setminus \{x_0\}, \preceq)$. Let $I = \{\, x : x \prec x_0 \,\}$, $F = \{\, y : x_0 \prec y \,\}$: if $x \in I$, $y \in F$ then $x \prec y$ and hence $x \triangleleft_i y$.

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Define linearizations $\trianglelefteq_0'$ and $\trianglelefteq_0''$ of $(P, \preceq)$ by:

$$I \lhd_0' \{x_0\} \lhd_0' P \setminus \{x_0\} \cup I;$$

$$P \setminus \{x_0\} \cup F \lhd_0'' \{x_0\} \lhd_0'' F;$$

(each piece ordered by $\trianglelefteq_0$).
$\trianglelefteq_0'$ and $\trianglelefteq_0''$ take care of the incomparabilities between x_0 and the elements of P. Moreover, on each pair in $P \setminus \{x_0\}$ at least one of $\trianglelefteq_0'$ and $\trianglelefteq_0''$ preserves the order of $\trianglelefteq_0$.

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$\{\trianglelefteq_0', \trianglelefteq_0'', \trianglelefteq_1', \dots, \trianglelefteq_{m-1}'\}$ realize $(P, \preceq)$

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The key step of the previous proof is obtaining B_i for $i < m$, which exist by a separation principle.

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Within RCA_0*, the following are equivalent:*

1. WKL_0*;*
2. *if* $(P, \preceq)$ *is a poset and* $I, F \subseteq P$ *are such that* $\forall x \in I \, \forall y \in F \, y \not\preceq x$ *then there exists a downward closed set* $B \subseteq P$ *such that* $I \subseteq B$ *and* $B \cap F = \emptyset$*.*

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Applying (2) to the disjoint union of $(P, \trianglelefteq_0)$, ..., $(P, \trianglelefteq_{m-1})$ we get the B_i's.

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However to prove the reversal of the lemma we used a poset with plenty of infinite antichains (in I, F and $P \setminus I \cup F$).
Here we are applying (2) to a poset of width m, so it is possible that the full strength of WKL_0 is not needed.

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Lemma (Linear Separation Lemma LSL)

RCA$_0$ *proves that if* $(P, \trianglelefteq)$ *is a linear order and* $I, F \subseteq P$ *are such that* $\forall x \in I \, \forall y \in F \; y \ntrianglelefteq x$ *then there exists a downward closed set* $B \subseteq P$ *such that* $I \subseteq B$ *and* $B \cap F = \emptyset$.

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The proof is non-uniform:

- if there exists $z \in P$ such that $\forall x \in I\, \forall y \in F\, x \triangleleft z \triangleleft y$ then we can take $B = \{ u \in P : u \trianglelefteq z \}$;
- if z does not exists then for every $u \in P$ then
$$\exists x \in I\, u \trianglelefteq x \iff \forall y \in F\, u \triangleleft y$$
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To prove DB$_\mathsf{p}$ we use LSL m times, asking m $\mathbf{\Sigma}^0_2$ questions.
By bounded $\mathbf{\Sigma}^0_2$-comprehension, i.e. $\mathbf{\Sigma}^0_2$-induction, we know which of the two definitions of B_i we must use (in the first case the least $\mathbf{\Pi}^0_1$ principle finds the appropriate z).

The upper bound for DB_p

Theorem

The disjunction of WKL_0 *and* $\mathsf{RCA}_0 + \mathsf{I}\Sigma^0_2$ *proves* $\mathsf{DB}_\mathbf{p}$.

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In the literature there are a few statements known to be equivalent to $\mathsf{WKL}_0 \vee \mathsf{I}\boldsymbol{\Sigma}^0_2$:

- the existence for all n of n-fold iterates of continuous mappings of the closed unit interval into itself (Friedman-Simpson-Yu, 1993);
- every complete consistent theory with countably many types and whose types have the pairwise full amalgamation property has a saturated model (Belanger, 2015) [the equivalence is over $\mathsf{RCA}_0 + \mathsf{B}\boldsymbol{\Sigma}^0_2$, and $\mathsf{B}\boldsymbol{\Sigma}^0_2$ cannot be removed];
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So far we haven't been able to exploit these results or mimic their proofs to show that $\mathsf{DB_p}$ is equivalent to $\mathsf{WKL}_0 \vee \mathsf{I}\boldsymbol{\Sigma}^0_2$.

Thank you for your attention!