Reverse Mathematics and Hindman's theorem

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This talk is dedicated to Jeff Hirst.

As the first talk at Hirstfest, I thought I would start with the beginning: a topic from his PhD thesis.

I will conclude with some remarks on his publications and collaborators, several of whom are here today for Hirstfest.

MathSciNet currently lists 52 publications from Jeff Hirst, with 32 total coauthors.

Introduction

I will attempt a comprehensive picture of work related to Hindman's theorem from the last 40+ years.

Numerous logicians have worked successfully and unsuccessfully on this theorem.

Several key results are due to Jeff Hirst and coauthors.

There is much more in the extensive literature than I could discuss in this talk. I apologize for any unintentional omissions.

Hindman's theorem

Hindman proved the following theorem in 1974.

Theorem (HT: Hindman's Finite Sums Theorem)

If $\mathbb{N}$ is colored with finitely many colors, there is an infinite $A \subseteq \mathbb{N}$ such that, for every finite nonempty $F \subseteq A$, the number $\sum F$ has the same color.

The conclusion states that the set $\mathrm{FS}(A)$ of finite sums of distinct elements of A is monochromatic.

Early history

The finitary analog of Hindman's theorem, in which the set A must be arbitrarily large rather than finite, is known as **Folkman's theorem**.

This analog was proved independently by several individuals ca. 1968–1970, and named after Folkman by Graham, Rothschild, and Spencer.

Around this time, HT was stated as a conjecture by Graham and Rothschild.

This early history is described by Hindman (2005).

Galvin's question

In the early 1970s, Galvin realized there is a straightforward proof of HT from the existence of an almost translation invariant ultrafilter.

- An **ultrafilter** on $\mathbb{N}$ is a maximal subset $\mathcal{F}$ of $P(\mathbb{N})$ such that the intersection of any nonempty family from $\mathcal{F}$ is nonempty.
- An ultrafilter $\mathcal{F}$ is **almost translation invariant (a.t.i.)** if, whenever $A \in \mathcal{F}$, the set $\{x \in \mathbb{N} : A - x \in \mathcal{F}\}$ is in $\mathcal{F}$.
  Here $A - x = \{y : y + x \in A\}$.

Galvin posed the question of whether there is an a.t.i. ultrafilter and Erdös circulated the question.

Hindman's proof

Hindman was unable to prove that a.t.i. ultrafilters exist in ZFC.

He *was* able to prove that HT and the continuum hypothesis together imply a.t.i. ultrafilters exist.

Hindman then obtained a direct combinatorial proof of HT.

> *"If the reader has a graduate student that she wants to punish, she should make him read and understand that original proof..."* (Hindman 2005)

Hirst's thesis

In his thesis *Combinatorics in Subsystems of Second-Order Arithmetic*, Hirst made important contributions related to several theorems:

- Hall's marriage theorem
- Ramsey's theorem
- Hindman's theorem

Hirst's thesis

The work on Hindman's theorem had several parts:

- A collaboration of Blass, Hirst, and Simpson the provided the still-best upper bounds for the strength of Hindman's theorem.
- A reformulation of Hindman's theorem in terms of ideals on Boolean rings.
- A study of Milliken's theorem, a generalization equivalent to an iterated version of Hindman's theorem.

Subsystems of second order arithmetic

- RCA_0 is the standard base system including Δ^0_1 comprehension and Σ^0_1 induction.
- ACA_0 is RCA_0 plus an axiom stating the Turing jump $\mathrm{TJ}(A)$ exists for every $A \subseteq \mathbb{N}$.
- ACA_0^+ is RCA_0 plus an axiom stating that the Turing jump may be iterated along $\mathbb{N}$. Roughly: $\mathrm{TJ}^{(\omega)}(A)$ exists for all $A \subseteq \mathbb{N}$.

Bounds on the strength of HT

Let HT_2 be the restriction of HT to two colors.

Theorem (Blass, Hirst, and Simpson 1987)

- HT *is provable in* ACA_0^+.
- HT_2 *implies* ACA_0 *over* RCA_0.

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Theorem (Blass, Hirst, and Simpson 1987)

- *Every computable instance* C *of* HT_2 *has a solution computable from* $C^{(\omega+1)}$.
- *There is a computable instance* C *of* HT_2 *such that every solution to* C *computes* $\emptyset'$.
- *There is a computable instance of* HT_2 *with no* Δ^0_2 *solution.*

The reversal

We sketch the construction of a computable 2-coloring of $\mathbb{N}$ so that every solution to HT computes $\emptyset'$.

Choose e so that $\emptyset' = W_e = K$, and for each t let $K(s) = W_{e,s}$.

For $n = 2^{n_1} + \cdots + 2^{n_k}$:

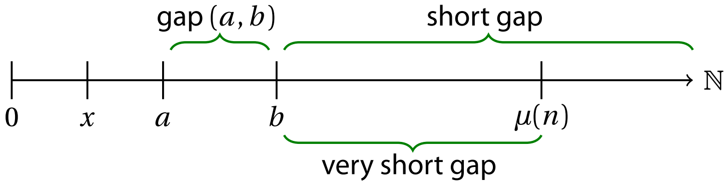
- View as a code for the sequence $s(n) = (n_1, \ldots, n_k)$.
- Let $\lambda(n) = n_1$ and $\mu(n) = n_k$.
- The pairs (n_i, n_{i+1}) are the *gaps* of n.

Note: if $\mu(n) < \lambda(m)$ then the gaps of $n+m$ are the gaps of n, the gaps of m, and the new gap $(\mu(n), \lambda(m))$.

Gaps

A gap (a, b) of a number n is defined to be:

- *short* if there is an $x \leq a$ such that $x \in K$ and $x \notin K(b)$.
- *very short* if there is an $x \leq a$ such that $x \in K(\mu(n))$ and $x \notin K(b)$.



Coloring

Define

- $\mathrm{SG}(n)$ is the number of short gaps of n.
- $\mathrm{VSG}(n)$ is the number of very short gaps of n.

Because $\mathrm{VSG}(n)$ is computable from n, the following coloring c is computable:

$$c(n) = \mathrm{VSG}(n) \mod 2$$

The reversal

Given any solution A to HT for the coloring c, we can compute another solution $B \subseteq \mathrm{FS}(A)$ *with apartness*: so that $\mu(m) < \lambda(n)$ whenever $m < n$ are in B.

Claim 1. For every $m \in \mathrm{FS}(B)$, $\mathrm{SG}(m)$ is even.

This follows from a parity argument and some basic computability analysis of the coloring.

The reversal

Claim 2. If $m < n$ are in B and $x \leq \mu(m)$ then $x \in K$ if and only if $x \in K(\lambda(n))$.

The key point is that, for this to fail, the gap $(\mu(m), \lambda(n))$ would be short. But this would mean

$$\mathrm{SG}(m+n) = \mathrm{SG}(m) + 1 + \mathrm{SG}(n),$$

which is impossible because the three SG terms are all even.

Alternative proofs

There are three additional proofs of HT, each of which seems to require stronger systems.

- A simplified inductive proof due to Baumgartner.
- A proof using ultrafilters, originally due to Galvin and Glazer.
- A proof of HT as a consequence of the Auslander–Ellis theorem, which is proved using higher-order methods.

Each of these has been studied in the context of reverse mathematics.

Baumgartner's proof

Baumgartner (1974) produced a simpler, still inductive proof of HT.

Blass, Hirst, and Simpson showed this proof can be formalized in the system Π^1_2-TI_0, which is far stronger than ACA_0^+.

Topological dynamics

Blass, Hirst, and Simpson also study a proof of a theorem of topological dynamics.

We view a compact metric space X and a continuous $f : X \to X$ as a *compact dynamical system*.

- A point z in a compact dynamical system is *uniformly recurrent* if for all $\varepsilon > 0$ the set $\{n : d(f^n(x), x) < \varepsilon\}$ has bounded gaps.
- Two points y, z are *proximal* if for all $\varepsilon > 0$ there are infinitely many n such that $d(f^n(y), f^n(z)) < \varepsilon$.

The Auslander–Ellis theorem

Theorem (AET: Auslander and Ellis)

If (X, f) is a compact dynamical system, for every $y \in X$ there is a $z \in X$ such that y is proximal to z and z is uniformly recurrent.

In the 1970s, Furstenberg showed that HT has a straightforward proof using AET as the key lemma.

The textbook proof of AET is higher-order and not clearly formalizable in second order arithmetic.

Blass, Hirst, and Simpson showed, using their analysis of HT, that AET is provable in ACA_0^+, and hence does not require set-theoretic methods.

The Galvin–Glazer proof

In 1975, by applying knowledge from the theory of topological semigroups, Glazer showed that almost translation invariant ultrafilters exist.

The proof is based on the fact that the set $\beta\mathbb{N}$ of ultrafilters on $\mathbb{N}$ is a topological semigroup under a particular operation.

Glazer was familiar with longstanding results about topological semigroups that imply the existence of idempotent elements. In this case, those will be a.t.i. ultrafilters.

Read directly, this proof uses fourth order objects: closed sets of ultrafilters on $\mathbb{N}$.

Ultrafilters and reverse mathematics

Ultrafilters on $\mathbb{N}$ are third-order objects, which cannot be directly represented in second order arithmetic.

There are two natural paths forward:

- Stay inside second order arithmetic: study countable subsets of ultrafilters rather than actual ultrafilters.
- Study ultrafilters using fragments of third order arithmetic.

Both of these paths have been explored.

Countable subalgebras

Hirst (2004) explored the possibility of miniaturizing the ultrafilter proof of HT into second order arithmetic.

The key idea was to work with countable subalgebras of $\mathcal{P}(\mathbb{N})$. These subalgebras can be easily formalized in ACA_0.

Countable subalgebras

In this context, Hirst obtained an equivalence between IHT and the existence of certain ultrafilters,

Theorem (Hirst 2004)

The following are equivalent over RCA_0:

- *Every countable downward translation algebra has an almost translation invariant ultrafilter.*
- Iterated Hindman's Theorem (IHT_2): *Given a sequence* C_i *of* 2*-colorings of* $\mathbb{N}$*, there is an infinite sequence* $\langle x_i \rangle$ *of numbers such that, for each* i, $\mathrm{FS}(\{x_j : j > i\})$ *is monochromatic for* C_j.

A lightweight ultrafilter proof

Towsner (2011) showed that the standard Galvin–Glazer proof can be miniaturized in a way that can be formalized in second order arithmetic.

The key idea is to replace applications of Zorn's lemma with transfinite inductions.

This argument lies, in a sense, between the Galvin–Glazer proof and the Baumgartner proof.

A simpler inductive proof

Towsner (2012) presented a simplified inductive proof of HT in ACA_0^+. This can be viewed as a culmination of miniaturizing the Galvin–Glazer proof into Z_2, extending the previous result.

Towsner (2012) and Liao (2024) improved the lower bound on the noncomputability of solutions.

Theorem (Towsner 2012)

There is a computable instance of HT_2 *with no* Σ^0_2 *solution.*

Theorem (Liao 2024)

There is a computable instance of HT_4 *with no* Π^0_3 *solution.*

Beyond second order arithmetic

Kreuzer (2012) and Towsner (2014) and independently studied the second option for handling ultrafilters: move beyond second order arithmetic.[1]

Their work was foreshadowed in previous work of Enyat (2006) on generic ultrafilters over models of ACA_0.

[1]Both authors submitted preprints to the ArXiV in September 2011.

Ultrafilters in higher-order arithmetic

Kreuzer worked directly in higher-order arithmetic using systems such as ACA_0^ω that are conservative over their corresponding second order systems.

- $(\mathcal{U})$ states that a nonprincipal ultrafilter exists
- $(\mathcal{U}_{\mathrm{idem}})$ states that an idempotent ultrafilter exists
- (μ) is a standard axiom which allows for the uniform computation of Turing jumps in higher-order arithmetic

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Theorem (Kreuzer 2012)

$\mathsf{ACA}_0^\omega + (\mathcal{U})$ *is* Π^1_2 *conservative over* ACA_0^ω.

Theorem (Kreuzer 2015)

$\mathsf{ACA}_0^\omega + (\mu) + \mathsf{IHT} + (\mathcal{U}_{idem})$ *is* Π^1_2 *conservative over* $\mathsf{ACA}_0^\omega + \mathsf{IHT}$.

Conservativity over second order systems

Towsner (2014) studied systems that extend second order arithmetic with:

- An added third-order predicate $\mathcal{U}$ for a subset of $P(\mathbb{N})$ and some associated syntax.
- An axiom $\exists\mathcal{U}$ stating that $\mathcal{U}$ defines a nonprincipal ultrafilter on $\mathbb{N}$.

Theorem (Towsner 2014)

Let $T \in \{\mathsf{ACA}_0, \mathsf{ATR}_0, \Pi^1_1\text{-}\mathsf{CA}_0\}$. *Then* $T + \exists\mathcal{U}$ *is conservative over* T.

Conservativity over second order systems

Montálban and Shore answered a question of Towsner by showing that the existence of an idempotent ultrafilter is conservative over a number of systems.

Theorem (Montálban and Shore 2018)

- *The assertion* $\exists\mathcal{U}_{\mathrm{idem}}$ *is conservative over* $\mathsf{ACA}_0 + \mathsf{IHT}$, ATR_0, Π^1_1-CA_0 *and* Π^1_2-CA_0.
- *The Milliken–Taylor theorem* $(\forall k)\mathsf{MT}(k)$ *is provable in* $(\mathsf{ACA}_0^+)'$.

The second result is related to the result of Hirst (2004) showing IHT is equivalent to $\mathsf{MT}(k)$ for each $k \geq 3$.

Rado's path decomposition theorem

Cholak and coauthors observed that conservation results can be applied to an ultrafilter-based proof to obtain a proof in second order arithmetic.

Observation (Cholak, Igusa, Patey, Soskova & Turetsky 2019)

The Rado Path Decomposition theorem is provable in ACA_0.

The authors go on to show that the theorem is equivalent to ACA_0 over RCA_0.

Nonprincipal ultrafilters

Eastaugh (2024) studied the strength of the existence of a nonprincipal ultrafilter on a countable atomic algebra.

Theorem (Eastaugh 2024)

The following are equivalent over RCA_0:

- ACA_0.
- *For every infinite* $V \subseteq \mathbb{N}$ *and every atomic countable algebra* A *over* V, *there exists a non-principal ultrafilter* U *on* A.

The theorem is part of an analysis of Arrow's voting theorem.

Theorem (Summary of the literature)

The following are equivalent over RCA_0.

Each is provable in ACA_0^+ *and implies* ACA_0 *over* RCA_0.

- IHT: *For every sequence of finite colorings* $\langle C_i \mid i \in \mathbb{N}\rangle$ *there is an increasing sequence* $\langle x_i \in \mathbb{N} \mid i \in \mathbb{N}\rangle$ *such that for every* $j \in \mathbb{N}$ *the set* $\{x_i \mid i > j\}$ *satisfies Hindman's Theorem for* C_i.
- IHT *restricted to* 2*-colorings.*
- *Every countable downward translation algebra has an almost downward translation invariant ultrafilter.*
- *The Auslander–Ellis theorem.*
- *For each* $k \geq 3$, *the principle* $\mathsf{MT}(k)$.

Restrictions

One way to restrict Hindman's theorem is to ask for only certain finite sums to be monochromatic.

- $\mathsf{HT}_k^{\leq n}$ states that, given a k-coloring of $\mathbb{N}$, there is an infinite set A so that all sums of $\leq n$ elements of A have the same color.
- $\mathsf{HT}_k^{=n}$ states that, given a k-coloring of $\mathbb{N}$, there is an infinite set A so that all sums of n elements of A have the same color.

A set A has b-*apartness* if whenever $x < y \in A$ and x, y are interpreted as sequences base b, we have $\mu(x) < \lambda(y)$.

Restrictions: bounded sums

Carlucci (2021) provides an overview of work on restrictions of HT.

Theorem (Dzhafarov, Jockusch, Solomon & Westrick 2017)

$\mathsf{HT}^{\leq 3}$ *implies* ACA_0 *over* RCA_0.

Theorem (Carlucci, Kołodziejczyk, Lepore & Zdanowski 2020)

- $\mathsf{HT}^{\leq 2}_4$ *implies* ACA_0 *over* RCA_0.
- $\mathsf{HT}^{=3}$ *with apartness implies* ACA_0 *over* RCA_0.

Restriction: exact sums

For sums of exactly 2 or 3 elements, HT is related to the rainbow Ramsey theorem RRT and the increasing polarized Ramsey theorem IPT of Dzhafarov and Hirst (2011)

Theorem (Csima, Dzhafarov, Hirschfeldt, Jockusch, Solomon & Westrick 2019)

Over RCA_0, $\mathsf{HT}_2^{=2}$ *implies the rainbow Ramsey theorem* RRT_2^2.

Theorem (Carlucci, Kołodziejczyk, Lepore & Zdanowski 2020)

Over RCA_0, $\mathsf{HT}_2^{=2}$ *with apartness implies* IPT_2^2.

The role of apartness is interesting, as RRT_2^2 is weaker than IPT_2^2.

Variation: thin solutions

Hirschfeldt and Reitzes (2022) studied a thin version of HT.

Given a coloring $c : \mathbb{N} \to \mathbb{N}$, a set B is *thin* if at least one color is omitted from $c(B)$.

The principle thin-HT states that given $c : \mathbb{N} \to \mathbb{N}$, there is an infinite set A such that $\mathrm{FS}(A)$ is thin for c.

Theorem (Hirschfeldt and Reitzes 2022)

- *There is a computable instance of thin-*HT *such that every solution computes* $\emptyset'$.
- *thin-*HT *implies* ACA_0 *over* $\mathsf{RCA}_0 + \mathsf{I}\Sigma^0_2$.
- RRT^2_2 *is Weihrauch reducible to a strengthened version of thin-*$\mathsf{HT}^{=2}$.

Hindman's theorem

Much like Ramsey's theorem, Hindman's theorem has been a source of numerous threads in reverse mathematics.

- The precise strength of HT and IHT remain longstanding open problems.
- The various proofs of HT have inspired their own work on formalizing ultrafilter methods into arithmetic.
- Restrictions of HT also lead to interesting combinatorial results and distinctions.

Reflections on my work with Jeff Hirst

Jeff and I have been coauthors on five papers, including combinatorics, proof theory, and Weihrauch reducibility.

I had the opportunity to work with Jeff at Appalachian State in 2005–2006 as the first of a series of “postdocs” there.

It has been a pleasure to work with Jeff as a colleague and friend throughout my career.

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